

Semi-parallelism of normal Jacobi operator for Hopf hypersurfaces in complex two-plane Grassmannians

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Abstract. It is proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose normal Jacobi operator is semi-parallel, if the principal curvature of the Reeb vector field is non-vanishing and the component of the Reeb vector field in the maximal quaternionic subbundle \mathfrak{D} or its orthogonal complement \mathfrak{D}^\perp is invariant by the shape operator.

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1 Introduction

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is the set of all 2-dimensional linear subspaces in \mathbb{C}^{m+2} . It is a symmetric space and is equipped with both a Kaehler structure J and a quaternionic Kaehler structure J with a canonical local basis $\{J_1, J_2, J_3\}$, which does not contain J .

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, N a unit normal vector field of M and A the shape operator of M with respect to N . The Reeb vector field of M is the structure vector field given by $\xi = -JN$. Apart from the Reeb vector field, there are three more vector fields given by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Consequently, we have two distributions on M given by $[\xi] = \text{Span}\{\xi\}$ and $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. We denote by \mathfrak{D} the orthogonal complement of the distribution \mathfrak{D}^\perp such that $T_p M = \mathfrak{D}_p \oplus \mathfrak{D}_p^\perp$, for each point $p \in M$.

An important geometric condition for real hypersurfaces is the invariantness of the distributions $[\xi]$ and \mathfrak{D}^\perp under the action of the shape operator. Under this condition, using a result due to Alekseevskii [1], Berndt and Suh classified the real hypersurfaces in the following:

Theorem 1.1 (Theorem 1, [4]) *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both the distributions $[\xi]$ and \mathfrak{D}^\perp are invariant under the shape operator of M if and only if either*

- *M is of type (A), that is M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
- *M is of type (B), that is m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{2n+2})$.*

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a Hopf hypersurface if the Reeb vector field ξ is principal, that is $A\xi = \alpha\xi$, where $\alpha = g(A\xi, \xi)$ is the corresponding principal curvature to ξ . In such a case the integral curves of the Reeb vector field ξ are geodesics

(Berndt and Suh [5]). Of course, all of hypersurfaces in $G_2(\mathbb{C}^{m+2})$ mentioned in Theorem 1.1 are Hopf hypersurfaces.

In [2], Berndt introduced the notion of *normal Jacobi operator*

$$\overline{R}_N(X) = \overline{R}(X, N)N \in \text{End}(T_x M), \quad x \in M,$$

for a real hypersurface M in quaternionic projective spaces $\mathbb{H}P^m$ and in quaternionic hyperbolic spaces $\mathbb{H}H^m$, where \overline{R} is the curvature tensor of the ambient space. He also proved the equivalence of the commutation of \overline{R}_N with the shape operator A with the fact that the distributions \mathfrak{D} and \mathfrak{D}^\perp are invariant under the shape operator A .

The classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, whose normal Jacobi operator \overline{R}_N satisfies certain geometric conditions, is one of great importance in the area of Differential Geometry. In [15], Perez et. al. proved that \mathfrak{D}^\perp -invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, whose normal Jacobi operator commutes with both the structure tensor φ and the shape operator A are locally congruent to one of type (A). Recently in [11], Jeong, Suh and the second author considered Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ which satisfy the following two commuting conditions

$$\varphi A \overline{R}_N X = \overline{R}_N \varphi A X, \quad X \in TM \quad \text{and} \quad A \varphi \varphi_1 X = \varphi \varphi_1 A X, \quad X \in \mathfrak{D}^\perp;$$

and proved that such real hypersurfaces are locally congruent to one of type (A). The first condition is equivalent to $(\mathcal{L}_\xi \overline{R}_N)X = (\nabla_\xi \overline{R}_N)X$.

There are many interesting results concerning the non-existence of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ under certain geometric conditions on the normal Jacobi operator. In [7], Jeong and Suh examined cases of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, when the normal Jacobi operator is Lie ξ -parallel, that is $\mathcal{L}_\xi \overline{R}_N = 0$. More precisely, they proved the non-existence of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \overline{R}_N = 0$ and one of the conditions $\xi \in \mathfrak{D}^\perp$ and $\xi \in \mathfrak{D}$. They also proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \overline{R}_N = 0$ and commuting shape operator on the distribution \mathfrak{D}^\perp .

In [9], it was proved that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ does not exist if the normal Jacobi operator is Lie parallel and the integral curves of \mathfrak{D} - and \mathfrak{D}^\perp - components of the Reeb vector field are totally geodesic. In [13], Machado et. al. proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ whose normal Jacobi operator is of Codazzi type (that is, $(\nabla_X \overline{R}_N)Y = (\nabla_Y \overline{R}_N)X$ for any $X, Y \in TM$) and \mathfrak{D} - or \mathfrak{D}^\perp -component of ξ is invariant by the shape operator. In [8], Jeong et. al. proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator, that is $\nabla_X \overline{R}_N = 0$. In [10], the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ whose normal Jacobi operator is $([\xi] \cup \mathfrak{D}^\perp)$ -parallel, which is a weaker condition than the previous one, was proved.

A tensor field P of type $(1, s)$ on a Riemannian manifold is said to be *semi-parallel* if $R \cdot P = 0$, where R is the curvature tensor of the manifold and acts as a derivation on P [6]. In the geometry of real hypersurfaces in complex space form the following results concerning the semi-parallelism conditions have been proved. In [16], Perez and Santos proved that there exist no real hypersurfaces in complex projective space CP^n , $n \geq 3$, with semi-parallel structure Jacobi operator (that is $R \cdot R_\xi = 0$, where $R_\xi = R(\cdot, \xi)\xi$ and ξ is the structure vector field). Later, Cho and Kimura [6] generalized this work and proved that there do not

exist real hypersurfaces in complex space forms equipped with semi-parallel structure Jacobi operator. Finally, Niebergall and Ryan in [14] studied real hypersurfaces in complex space forms equipped with the semi-parallel shape operator A .

Motivated by these studies the following question is raised naturally:

Problem 1.2 Do there exist real hypesurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose normal Jacobi operator, structure Jacobi operator or shape operator is semi-parallel?

In the present paper we give the answer partially and prove the following:

Theorem 1.3 *There does not exist any connected Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, equipped with semi-parallel normal Jacobi operator, if $\alpha \neq 0$ and \mathfrak{D} - or \mathfrak{D}^\perp -component of the Reeb vector field ξ is invariant by the shape operator A .*

The paper is organized as follows. In section 2, we give a brief description of complex two plane Grassmanians. In section 3 basic relations for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ are presented. Section 4 contains some key results for further use. Finally, in section 5, we give the proof of Theorem 1.3.

2 Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is the Grassmann manifold of all complex 2-dimensional linear subspaces in \mathbb{C}^{m+2} . The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Thus $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K , which can be equipped with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by \mathfrak{g} and \mathfrak{l} the Lie algebra of G and K , respectively. Let \mathfrak{m} be the orthogonal complement of \mathfrak{l} in \mathfrak{g} with respect to the Cartan-Killing form B of \mathfrak{g} . Then $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ is an $Ad(K)$ -invariant reductive decomposition of \mathfrak{g} . we put $o = eK$ and identify $T_o G_2(\mathbb{C}^{m+2})$ with \mathfrak{m} in the usual manner. Since B is negative definite on \mathfrak{g} , therefore the restriction $(-B)|_{\mathfrak{m} \times \mathfrak{m}}$ yields a positive definite inner product on \mathfrak{m} . By $Ad(K)$ -invariance of B this inner product can be extended to a G -invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this manner $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous symmetric space. For computational reasons we normalize the Riemannian metric g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ becomes 8.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the 2-dimensional complex projective space \mathbb{CP}^2 with constant holomorphic sectional curvature 8. When $m = 2$, the isomorphism $\text{Spin}(6) \simeq SU(4)$ provides an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented 2-dimensional linear subspaces of \mathbb{R}^6 . Therefore, we usually assume that $m \geq 3$.

The Lie algebra \mathfrak{l} has the direct sum decomposition $\mathfrak{l} = \mathfrak{su}(m) \oplus \mathfrak{su}(m) \oplus \mathfrak{R}$, where \mathfrak{R} is the center of \mathfrak{l} . Regarding \mathfrak{l} as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center \mathfrak{R} induces a Kaehler structure J and the $\mathfrak{su}(2)$ -part induces a quaternionic Kaehler structure \mathfrak{J} on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in \mathfrak{J} , then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\text{tr}(JJ_\nu) = 0$.

A canonical local basis $\{J_1, J_2, J_3\}$ of \mathfrak{J} consists of three local almost Hermitian structures J_ν in \mathfrak{J} such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index is taken modulo 3. Since \mathfrak{J} is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of \mathfrak{J} three local 1-forms q_1, q_2, q_3 , such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \quad (2.1)$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemann curvature tensor \bar{R} of $G_2(\mathbb{C}^{m+2})$ is locally given by [3]

$$\begin{aligned} \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y \\ &+ g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &+ \sum_{\nu=1}^3 \{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\} \\ &+ \sum_{\nu=1}^3 \{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\} \end{aligned} \quad (2.2)$$

for all vector fields X, Y, Z on $G_2(\mathbb{C}^{m+2})$, where $\{J_1, J_2, J_3\}$ is any canonical local basis of \mathfrak{J} . This expression involves the Riemannian curvature tensor of S^{4m} , $\mathbb{C}P^{2m}$ and $\mathbb{H}P^m$.

3 Real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M is denoted by g and ∇ denotes the induced Riemannian connection of (M, g) . Let N be a local unit normal field of M and A the shape operator of M with respect to N .

Now let us put

$$JX = \varphi X + \eta(X)N, \quad J_\nu X = \varphi_\nu X + \eta_\nu(X)N \quad (3.1)$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The Kaehler structure J of $G_2(\mathbb{C}^{m+2})$ induces a local almost contact metric structure (φ, ξ, η, g) on M in the following way

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(X) = 1, \quad \varphi\xi = 0, \quad \eta(X) = g(X, \xi).$$

If M is orientable then ξ is globally defined and is the induced Reeb vector field on M . Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of \mathfrak{J} . Then each J_ν induces an almost contact metric structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$ on M . Locally, the orthogonal complement of the real span of ξ in TM is denoted by \mathfrak{H} and the orthogonal complement of the real span of ξ_1, ξ_2, ξ_3 in TM is denoted by \mathfrak{D} .

In view of (2.2), the Gauss equation is given by

$$\begin{aligned}
R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y \\
&+ g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z \\
&+ \sum_{\nu=1}^3 \{g(\varphi_\nu Y, Z)\varphi_\nu X - g(\varphi_\nu X, Z)\varphi_\nu Y - 2g(\varphi_\nu X, Y)\varphi_\nu Z\} \\
&+ \sum_{\nu=1}^3 \{g(\varphi_\nu \varphi Y, Z)\varphi_\nu \varphi X - g(\varphi_\nu \varphi X, Z)\varphi_\nu \varphi Y\} \\
&- \sum_{\nu=1}^3 \{\eta(Y)\eta_\nu(Z)\varphi_\nu \varphi X - \eta(X)\eta_\nu(Z)\varphi_\nu \varphi Y\} \\
&- \sum_{\nu=1}^3 \{\eta(X)g(\varphi_\nu \varphi Y, Z) - \eta(Y)g(\varphi_\nu \varphi X, Z)\}\xi_\nu \\
&+ g(AY, Z)AX - g(AX, Z)AY
\end{aligned} \tag{3.2}$$

where R denotes the curvature tensor of the real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

It is straightforward to verify the following identities

$$\begin{aligned}
\varphi_\nu \xi_{\nu+1} &= \xi_{\nu+2}, & \varphi_{\nu+1} \xi_\nu &= -\xi_{\nu+2}, \\
\varphi \xi_\nu &= \varphi_\nu \xi, & \eta_\nu(\varphi X) &= \eta(\varphi_\nu X), \\
\varphi_\nu \varphi_{\nu+1} X &= \varphi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\
\varphi_{\nu+1} \varphi_\nu X &= -\varphi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}.
\end{aligned} \tag{3.3}$$

In view of (3.1), (2.1) and (3.3), it is known that

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX,$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \varphi_\nu AX,$$

$$(\nabla_X \varphi_\nu)Y = -q_{\nu+1}(X)\varphi_{\nu+2}Y + q_{\nu+2}(X)\varphi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu.$$

Summing up these formulas, we also find the following

$$\begin{aligned}
\nabla_X(\varphi_\nu \xi) &= (\nabla_X \varphi_\nu)\xi + \varphi_\nu(\nabla_X \xi) \\
&= -q_{\nu+1}(X)\varphi_{\nu+2}\xi + q_{\nu+2}(X)\varphi_{\nu+1}\xi \\
&\quad + \eta_\nu(\xi)AX - g(AX, \xi)\xi_\nu + \varphi_\nu \varphi AX.
\end{aligned}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\varphi_\nu \varphi X = \varphi \varphi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu.$$

For more details we refer to [1], [3], [4] and [5].

4 Key Lemmas

We consider a connected, orientable, Hopf hypersurface M in $G_2(C^{m+2})$ with $\alpha \neq 0$ and semi-parallel normal Jacobi operator. The normal Jacobi operator \overline{R}_N for a real hypersurface M in $G_2(C^{m+2})$ is given by

$$\begin{aligned} \overline{R}_N(X) &= X + 3\eta(X)\xi + 3 \sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^3 \{ \eta_\nu(\xi) (\varphi_\nu \varphi X - \eta(X)\xi_\nu) - \eta_\nu(\varphi X)\varphi_\nu \xi \} \end{aligned} \quad (4.1)$$

for any vector field X tangent to M . Furthermore, semi-parallelism condition of it, that is $R(X, Y) \cdot \overline{R}_N = 0$, implies

$$R(X, Y)\overline{R}_NZ = \overline{R}_N(R(X, Y)Z) \quad (4.2)$$

for all vector fields X, Y, Z tangent to M .

Lemma 4.1 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ such that \mathfrak{D} - or \mathfrak{D}^\perp -component of ξ is invariant by the shape operator A and $\alpha \neq 0$. If the normal Jacobi operator is semi-parallel, then $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.*

Proof. Suppose that ξ is written as

$$\xi = \eta(U)U + \eta(\xi_1)\xi_1 + \eta(\xi_2)\xi_2 + \eta(\xi_3)\xi_3, \quad (4.3)$$

where U is a unit vector in \mathfrak{D} and $\eta(U) \neq 0$ and $\eta(\xi_\kappa) \neq 0$ for at least one $\kappa \in \{1, 2, 3\}$. Then relation (4.3) implies that

$$\varphi_\kappa \xi = \eta(U)\varphi_\kappa U + \eta(\xi_{\kappa+1})\xi_{\kappa+2} - \eta(\xi_{\kappa+2})\xi_{\kappa+1}. \quad (4.4)$$

From (4.1), we get

$$\overline{R}_N(\xi) = 4\xi + 4 \sum_{\nu=1}^3 \eta(\xi_\nu)\xi_\nu, \quad (4.5)$$

$$\overline{R}_N(\xi_\kappa) = 4\xi_\kappa + 4\eta(\xi_\kappa)\xi + 2\eta(\xi_{\kappa+1})\varphi_{\kappa+2}\xi - 2\eta(\xi_{\kappa+2})\varphi_{\kappa+1}\xi, \quad (4.6)$$

$$\overline{R}_N(\varphi_\kappa \xi) = 2\eta(\xi_{\kappa+1})\xi_{\kappa+2} - 2\eta(\xi_{\kappa+2})\xi_{\kappa+1}. \quad (4.7)$$

Since the normal Jacobi operator is semi-parallel, from (4.2) and (4.5), we get

$$\overline{R}_N(R(\xi, \xi_\kappa)\xi) = 4R(\xi, \xi_\kappa)\xi + 4 \sum_{\nu=1}^3 \eta(\xi_\nu)R(\xi, \xi_\kappa)\xi_\nu. \quad (4.8)$$

Since \mathfrak{D} - or \mathfrak{D}^\perp -component of ξ is assumed to be invariant by the shape operator A , we obtain

$$AU = \alpha U \quad \text{and} \quad A\xi_\kappa = \alpha \xi_\kappa, \quad \kappa \in \{1, 2, 3\}. \quad (4.9)$$

In view of (4.9), from relation (3.2) we get

$$R(\xi, \xi_\kappa)\xi = \alpha^2\eta(\xi_\kappa)\xi - \alpha^2\xi_\kappa + 2\eta(\xi_{\kappa+1})\varphi_{\kappa+2}\xi - 2\eta(\xi_{\kappa+2})\varphi_{\kappa+1}\xi. \quad (4.10)$$

Substituting (4.10) in (4.8), we lead to the following

$$\begin{aligned} 4 \sum_{\nu=1}^3 \eta(\xi_\nu)R(\xi, \xi_\kappa)\xi_\nu &= \alpha^2\eta(\xi_\kappa)\overline{R}_N(\xi) - \alpha^2\overline{R}_N(\xi_\kappa) \\ &\quad + 2\eta(\xi_{\kappa+1})\overline{R}_N(\varphi_{\kappa+2}\xi) - 2\eta(\xi_{\kappa+2})\overline{R}_N(\varphi_{\kappa+1}\xi) \\ &\quad - 4\alpha^2\eta(\xi_\kappa)\xi + 4\alpha^2\xi_\kappa \\ &\quad - 8\eta(\xi_{\kappa+1})\varphi_{\kappa+2}\xi + 8\eta(\xi_{\kappa+2})\varphi_{\kappa+1}\xi. \end{aligned} \quad (4.11)$$

Taking the inner product of (4.11) with U , in view of (4.6), (4.7) and (4.4) we obtain

$$\sum_{\nu=1}^3 \eta(\xi_\nu)g(R(\xi, \xi_\kappa)\xi_\nu, U) = -\alpha^2\eta(\xi_\kappa)\eta(U). \quad (4.12)$$

We calculate $R(\xi, \xi_\kappa)\xi_\nu$ from relation (3.2) taking into account (4.9) and then we take the inner product with U and we lead to the following relation

$$g(R(\xi, \xi_\kappa)\xi_\nu, U) = \alpha^2\eta_\kappa(\xi_\nu)\eta(U). \quad (4.13)$$

From (4.12) and (4.13) we get

$$\alpha^2\eta(\xi_\kappa)\eta(U) = 0, \quad \kappa \in \{1, 2, 3\},$$

which is a contradiction. ■

Now, we examine the case when the Reeb vector field ξ belongs to the distribution \mathfrak{D}^\perp . In fact, we have the following

Lemma 4.2 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ and $\alpha \neq 0$, with semi-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$ then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. Let $W \in \mathfrak{D}$ arbitrarily. In order to prove that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$, it suffices to prove that $g(AW, \xi_\kappa) = 0$, $\kappa = 1, 2, 3$. Since $\xi \in \mathfrak{D}^\perp$, we have that $JN \in \mathfrak{J}N$. Let J_1 be an almost Hermitian structure of \mathfrak{J} such that $JN = J_1N$. Then we obtain that $\xi = \xi_1$ and $\eta(\xi_2) = \eta(\xi_3) = 0$. Furthermore, $\varphi\xi_2 = -\xi_3$, $\varphi\xi_3 = \xi_2$ and $\varphi(\mathfrak{D}) \subset \mathfrak{D}$.

Due to the fact that M is a Hopf hypersurface, we have that $A\xi = \alpha\xi$ and so $g(AW, \xi) = g(AW, \xi_1) = 0$. Thus, it remains to prove that

$$g(AW, \xi_\kappa) = 0, \quad \kappa = 2, 3.$$

From (4.1), we obtain

$$\overline{R}_N(\xi) = 8\xi, \quad \overline{R}_N(W) = W - \varphi_1\varphi W. \quad (4.14)$$

Using (4.14) in (4.2) we get

$$8R(W, \xi)\xi = \overline{R}_N(R(W, \xi)\xi). \quad (4.15)$$

In view of $A\xi = \alpha\xi$, from (3.2), it follows that

$$R(W, \xi)\xi = W + \alpha AW - \varphi_1\varphi W. \quad (4.16)$$

Substituting (4.16) in (4.15) and taking into consideration (4.14) we lead to the following

$$8W + 8\alpha AW - 8\varphi_1\varphi W = \overline{R}_N(W) + \alpha\overline{R}_N(AW) - \overline{R}_N(\varphi_1\varphi W). \quad (4.17)$$

From (4.1) we also get

$$\overline{R}_N(AW) = AW + 2\eta_2(AW)\xi_2 + 2\eta_3(AW)\xi_3 - \varphi_1\varphi AW,$$

$$\overline{R}_N(\varphi_1\varphi W) = \varphi_1\varphi W - \varphi_1\varphi(\varphi_1\varphi W).$$

Substitution of the previous two relations in (4.17) gives

$$7W + 7\alpha AW - 6\varphi_1\varphi W = 2\alpha\eta_2(AW)\xi_2 + 2\alpha\eta_3(AW)\xi_3 + \varphi_1\varphi(\varphi_1\varphi W) - \alpha\varphi_1\varphi AW.$$

Taking the inner product of the last relation with ξ_κ , $\kappa = 2, 3$, and because of $\alpha \neq 0$ implies

$$\eta_\kappa(AW) = 0, \quad \kappa = 2, 3,$$

and this completes the proof. ■

Finally, in the case when the Reeb vector field ξ belongs to the distribution \mathfrak{D} , we refer to the following

Proposition 4.3 (Proposition 3.1, [12]) *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector ξ belongs to the distribution \mathfrak{D} , then the distribution \mathfrak{D} is invariant under the shape operator A of M , that is $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

5 Proof of Theorem 1.3

In the previous section, because of Lemma 4.2, Proposition 4.3 and Theorem 1.1, we lead to the conclusion that real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, under some additional assumptions, whose normal Jacobi operator is semi-parallel are locally congruent to real hypersurfaces of type (A) or (B). Now, we check if the normal Jacobi operator of such real hypersurfaces satisfies the semi-parallelism condition.

First, we recall the following proposition due to Berndt and Suh ([4]).

Proposition 5.1 (Proposition 3, [4]) *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$ and ξ is tangent to \mathfrak{D}^\perp . Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \frac{\pi}{2\sqrt{8}}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8} \cot(\sqrt{8}r), \quad \beta = \sqrt{2} \cot(\sqrt{2}r), \quad \lambda = -\sqrt{2} \tan(\sqrt{2}r), \quad \mu = 0,$$

with some $r \in (0, \frac{\pi}{\sqrt{8}})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}\xi_1 = \mathbb{R}JN = \text{Span}\{\xi\} = \text{Span}\{\xi_1\}, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \text{Span}\{\xi_2, \xi_3\}, \\ T_\lambda &= \{X/X \perp \mathbb{H}\xi, \quad JX = J_1X\}, \\ T_\mu &= \{X/X \perp \mathbb{H}\xi, \quad JX = -J_1X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex, quaternionic span of the structure vector field ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of the $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

In this case we have $\xi = \xi_1$. From (4.1) we obtain

$$\overline{R}_N(\xi) = 8\xi \quad \text{and} \quad \overline{R}_N(\xi_2) = 2\xi_2. \quad (5.1)$$

Since the normal Jacobi operator is semi-parallel, from (4.2) and the second relation of (5.1) we obtain:

$$2R(\xi_2, \xi)\xi_2 = \overline{R}_N(R(\xi_2, \xi)\xi_2), \quad (5.2)$$

Relation (3.2) for $X = \xi_2$, $Y = \xi$ and $Z = \xi_2$ taking into account the fact that $A\xi = \alpha\xi$ and $A\xi_2 = \beta\xi_2$ implies

$$R(\xi_2, \xi)\xi_2 = -(2 + \alpha\beta)\xi. \quad (5.3)$$

Substitution of relation (5.3) in (5.2) leads to

$$(2 + \alpha\beta)\xi = 0.$$

The last relation taking into account that $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\beta = \sqrt{2}\cot(\sqrt{2}r)$ implies

$$\cot^2(\sqrt{2}r) = 0,$$

which is a contradiction. So real hypersurfaces of type **(A)** do not have semi-parallel normal Jacobi operator.

Next we check that whether real hypersurfaces of type **(B)** are equipped with semi-parallel normal Jacobi operator. We recall the following proposition due to Berndt and Suh ([4]).

Proposition 5.2 (Proposition 2, [4]) *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$ and ξ is tangent to \mathfrak{D} . Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r),$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \text{Span}\{\xi\}, \\ T_\beta &= \mathfrak{J}J\xi = \text{Span}\{\xi_1, \xi_2, \xi_3\}, \\ T_\gamma &= \mathfrak{J}\xi = \text{Span}\{\varphi_1\xi, \varphi_2\xi, \varphi_3\xi\}, \\ T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

From (4.1) we obtain

$$\overline{R}_N(W) = W, \quad \overline{R}_N(\xi) = 4\xi \quad \text{and} \quad \overline{R}_N(\xi_\nu) = 4\xi_\nu, \quad \nu = 1, 2, 3, \quad (5.4)$$

where $W \in T_\lambda$. Due to the semi-parallelism of the normal Jacobi operator, from (4.2) and the first relation of (5.4) we get:

$$R(W, \xi)W = \overline{R}_N(R(W, \xi)W), \quad (5.5)$$

The Gauss equation (3.2) for $X = W$, $Y = \xi$ and $Z = W$, because of $A\xi = \alpha\xi$ and $AW = \lambda W$ implies

$$R(W, \xi)W = -(1 + \alpha\lambda)\xi + \sum_{\nu=1}^3 g(\varphi_\nu \varphi W, W)\xi_\nu. \quad (5.6)$$

Substituting (5.6) in (5.5) and taking into account relation (5.4), we lead to the following

$$[1 + \alpha\lambda]\xi - \sum_{\nu=1}^3 g(\varphi_\nu \varphi W, W)\xi_\nu = 0.$$

The inner product of the last relation with ξ and substitution of $\alpha = -2 \tan(2r)$ and $\lambda = \cot(r)$ yield

$$1 - 2 \tan(2r) \cot(r) = 0,$$

from which we obtain

$$3 + \tan^2(r) = 0,$$

which is a contradiction. So real hypersurfaces of type **(B)** do not admit semi-parallel normal Jacobi operator and this completes the proof. ■

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